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Narumi-Katayama Polynomial of Some Nano Structures

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Abstract

The Narumi-Katayama index is the first topological index defined by the product of some graph theoretical quantities. Let G be a simple graph. Narumi-Katayama index of G is defined as the product of the degrees of the vertices of G. In this paper, we define the Narumi-Katayama polynomial of G. Next, we investigate some properties of this polynomial for graphs and then, we obtain this polynomial for some composite graphs such as splice, link, join, composition and Cartesian product of two graphs. Finally, using our results, we compute this polynomial for some nanostructures such as dendrimers and the chain of fullerenes.

Keywords: *Narumi-Katayama polynomial, Coefficients of a polynomial, Nanostar dendrimers, Fullerenes.*

1. INTRODUCTION

During this paper, we suppose that G is a simple, connected graph. Specifically, let $G = (V(G), E(G))$ be a graph with vertex set $V(G) = \{v_1, \ldots, v_n\}$ of order n and the edge set $E(G)$. For graph theoretic terminology we follow [1]. We denote the degree of a vertex v in G by $d(v)$ or $d_G(v)$, which is the number of edges incident to v. An *r-regular graph* is a graph such that the degree of each vertex is r. A graph G is *complete* if there is an edge between every pair of the vertices of G, i.e. a graph G is called *complete* if any two different vertices of G are adjacent. A complete graph on n vertices is denoted by K_n. A graph is called *bipartite* if its vertex set can be partitioned into two subsets X and Y such that every edge of G has one endpoint in X and the other endpoint in Y. In the case which $|X| = n$ and $|Y| = m$, we denote the *complete bipartite graph* by $K_{n,m}$. A *path* P_n is a sequence of vertices $v_1, \ldots, v_n$ such that each v_i is adjacent to v_{i+1}, for $i = 1, \ldots, n-1$. The path P_n with one more edge v_nv_1 is called a *n-cycle*. A *topological index* is a number invariant under automorphisms of the graph under consideration. In [2] Narumi and Katayama considered the product of $d(v)$ over all degrees of vertices in G as "simple topological index". Then the papers, mostly used from the name "Narumi-Katayama index" for this index. So we use from it in this paper, too. Also we denote the Narumi-Katayama index by NK. Thus, if G is a graph, then $NK(G) = \Pi_{v\in V(G)} d_G(v)$. In [3, 4], the authors investigated some properties of this topological index, but the main mathematical properties of NK index was reported by Klein and Rosenfeld [5]. This paper makes a new start on research about mathematical properties and chemical meaning of NK index. We encourage the interested readers to consult [6-9] and references therein for computational techniques as well as mathematical properties of topological indices. Consider the graph G with the vertex set $V(G) = \{v_1, \ldots, v_n\}$. We define the Narumi-Katayama polynomial for this graph as follow:

$$NK(G,x)=\prod_{i=1}^{n}\left(x+d_G(v_i)\right)$$

Therefore, one can see that $NK(G,0) = NK(G)$. Thus, if we obtain a result on Narumi-Katayama polynomial, we can apply it for Narumi-Katayama index.

The discovery of C_{60} *bucky-ball*, which has a nanometer-scale hollow spherical structure in 1985 by Kroto and Smalley revealed a new form of existence of carbon element other than graphite, diamond and amorphous carbon [10]. Fullerenes are molecules in the form of cage-like polyhedra, consisting solely of carbon atoms. A molecular graph is a simple graph such that its vertices correspond to the atoms and the edges to the bonds. Note that hydrogen atoms are often omitted. Molecular descriptors play a significant role in chemistry, pharmacology, etc. Among them, topological indices have a prominent place [11]. One of the important classes of molecular graphs are nanostar dendrimers. Nanostar dendrimers are a class of polymeric materials. They are highly branched, mono disperse macromolecules. The structure of these materials has a great impact on their physical and chemical properties. As a result of their unique behavior, nanostar dendrimers are suitable for a wide range of biomedical and industrial applications [12]. Many papers are constructed to investigate the topological indices of nanostar dendrimers, for more details we refer to [13-15]. In this paper, we investigate the properties of Narumi-Katayama polynomial. To do this, we begin with the investigation of coefficients of this polynomial. Also we compute the Narumi-Katayama polynomial of splice and link of two graphs, then we using them to compute the Narumi-Katayama polynomial of a class of nanostar dendrimers and the chain of fullerenes.

2. MAIN RESULTS

We start this section by computing Narumi-Katayama polynomial of some known graphs as it appears in the following table.

Table 1. *Narumi-Katayama polynomial of some known graphs.*

Graph G	$NK(G,x)$
K_n	$(x + n - 1)^n$
P_n	$(x + 1)^2(x + 2)^{n-2}$
n-cycle	$(x + 2)^n$
$K_{n,m}$	$(x + n)^m(x + m)^n$
W_n (Wheel graph)	$(x + n - 1)(x + 3)^{n-1}$
Petersen Graph	$(x + 3)^{10}$

Let G be a graph with vertex set $V(G) = \{v_1, \ldots, v_n\}$. Add new vertices $\{u_1, \ldots, u_n\}$ together with new edges $\{v_iu_i : 1 \le i \le n\}$ to G. Then the resulting graph is called *sun* and we denote it by $sun(G)$. Also, if we add to G the new vertices $\{u_{1,1}, \ldots, u_{1,r}, \ldots, u_{n,1}, \ldots, u_{n,r}\}$ together with new edges $\{v_iu_{i,j} : 1 \le i \le n, 1 \le j \le r\}$, we denote the resulting graph by $sun(G, r)$. Thus, it is easy to see that $sun(G; 1) = sun(G)$.

Theorem 2.1. *If G is a graph such that $V(G) = \{v_1, \ldots, v_n\}$, then*

$$NK\left(sun(G,r),x\right)=\left(x+1\right)^{nr}\prod_{i=1}^{n}\left(x+d_G(v_i)+r\right).$$

In particular, if G is k-regular, then

$$NK\left(sun(G,r),x\right)=\left(x+1\right)^{nr}\left(x+k+r\right)^{n}.$$

Proof. Since the degree of each vertex $u_{i,j} \in V(sun(G,r))$, $1 \le i \le n$, $1 \le j \le r$, is equal to 1, we have:

$$NK\left(sun(G,r),x\right)=\prod_{i=1}^{n}\left(x+d_{sun(G,r)}(v_i)\right)\prod_{j=1}^{r}\left(x+d_{sun(G,r)}(u_{i,j})\right)$$

$$=\prod_{i=1}^{n}\left(x+d_G(v_i)+r\right)\prod_{j=1}^{r}\left(x+1\right)$$

$$=\left(x+1\right)^{nr}\prod_{i=1}^{n}\left(x+d_G(v_i)+r\right)$$

and the proof is completed. ■

A *caterpillar tree* or *caterpillar* is a tree in which all the vertices are within distance

1 of a central path. If each vertex of the central path P_n has r pendant edges, we denote this caterpillar by cat(r). Therefore, it is easy to see that cat(r) = sun(P_n, r). Now, by Theorem 2.1, we have the following corollary.

Corollary 2.2. For the graph cat(r) we have

$$NK\left(cat\left(r\right),x\right)=\left(x+1\right)^{nr}\left(x+r+1\right)^{2}\left(x+r+2\right)^{n-2}.$$

Proof. Since cat(r) = sun(P_n, r), we can write

$$\begin{aligned} NK\left(cat\left(r\right),x\right) &= NK\left(sun\left(P_n,r\right),x\right) \\ &= \left(x+1\right)^{nr}\prod_{i=1}^{n}\left(x+d_{P_n}(v_i)+r\right) \\ &= \left(x+1\right)^{nr}\left(x+r+1\right)^{2}\left(x+r+2\right)^{n-2}. \end{aligned}$$

■

Now, we investigate the coefficients of x in the Narumi-Katayama polynomial of a graph G with n vertices. It is obvious that the coefficient of x^n in $NK(G,x)$ is 1. Let U be a subset of $V(G)$. The truncated Narumi-Katayama index, $NK^{(U)}$, in [15] has been defined as

$$NK^{(U)}(G)=\prod_{v\in V(G)-U} d_G(v)$$

Similarly, we define the truncated Narumi-Katayama polynomial of a graph G as fallow:

$$NK^{(U)}(G,x)=\prod_{v\in V(G)-U}\left(x+d_G(v)\right)$$

The following theorem shows a relationship between the coefficients of the variable x in Narumi-Katayama polynomial and truncated Narumi-Katayama index.

Theorem 2.3. Let G be a graph and $NK(G,x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$. Then

$$a_i=\sum_{\substack{U\subseteq V(G)\\ |U|=n-i}} NK^{(U)}\left(G\right)$$

Proof. Let $d_1, \ldots, d_n$ be the degree sequence of the graph G. So

$$NK\left(G,x\right)=\left(x+d_1\right)\cdots\left(x+d_n\right)$$

In this polynomial, by taking i number of x among the prentices, we should choose n-i degree of other vertices among the other prentices. So, the coefficient of x^i is the sum of the degrees of n-i vertices of G. In each choose of n-i vertices of G, we count a $NK^{(U)}$ for an $U\subseteq V(G)$ such that $|U|=n-i$ and so $a_i=\sum_{\substack{U\subseteq V(G)\\ |U|=n-i}} NK^{(U)}\left(G\right)$. ■

Suppose that G and H are two graphs of order n and m, respectively. For given vertices $v_n \in V(G)$ and $u_1 \in V(H)$ a *splice* of G and H by vertices v_n and u_1, $(G \bullet H)(v_n, u_1)$, is defined by identifying the vertices v_n and u_1 in the union of G and H. Similarly, a *link* of G and H by vertices v_n and u_1 is defined as the graph $(G \sim H)(v_n, u_1)$ obtained by joining v_n and u_1 by an edge in the union of these graphs [16]. In the next theorem, we obtain the Narumi-Katayama polynomial of these two graph operations.

Theorem 2.4. If G and H are two graphs with n and m number of vertices, respectively, then

$$NK\left(\left(G\bullet H\right)\left(v_n,u_1\right),x\right)=\frac{\left(x+d_G(v_n)+d_H(u_1)\right)}{\left(x+d_G(v_n)\right)\left(x+d_H(u_1)\right)}\times NK\left(G,x\right)NK\left(H,x\right)$$

and

$$NK\left(\left(G\sim H\right)\left(v_n,u_1\right),x\right)=\frac{\left(x+d_G(v_n)+1\right)\left(x+d_H(u_1)+1\right)}{\left(x+d_G(v_n)\right)\left(x+d_H(u_1)\right)}\times NK\left(G,x\right)NK\left(H,x\right)$$

Proof. Let G and H be two graphs such that $V(G) = \{v_1, \ldots, v_n\}$ and $V(H) = \{u_1, \ldots, u_m\}$. We have

$$\begin{aligned} NK\left(\left(G\bullet H\right)\left(v_n,u_1\right),x\right) &= \prod_{i=1}^{n-1}\left(x+d_{(G\bullet H)(v_n,u_1)}(v_i)\right) \\ &\times\prod_{j=2}^{m}\left(x+d_{(G\bullet H)(v_n,u_1)}(u_j)\right)\times\left(x+d_G(v_n)+d_H(u_1)\right) \\ &= \prod_{i=1}^{n-1}\left(x+d_G(v_i)\right)\prod_{j=2}^{m}\left(x+d_H(u_j)\right) \\ &\times\left(x+d_G(v_n)+d_H(u_1)\right) \end{aligned}$$

$$=\frac{\left(x+d_G(v_n)+d_H(u_1)\right)}{\left(x+d_G(v_n)\right)\left(x+d_H(u_1)\right)}\times NK\left(G,x\right)NK\left(H,x\right)$$

Similarly

$$NK\left(\left(G\sim H\right)\left(v_n,u_1\right),x\right)=\prod_{i=1}^{n}\left(x+d_{(G\sim H)(v_n,u_1)}(v_i)\right)$$
$$\times\prod_{j=1}^{m}\left(x+d_{(G\sim H)(v_n,u_1)}(u_j)\right)$$
$$=\prod_{i=1}^{n-1}\left(x+d_G(v_i)\right)\prod_{j=2}^{m}\left(x+d_H(u_j)\right)$$
$$\times\left(x+d_G(v_n)+1\right)\left(x+d_H(u_1)+1\right)$$
$$=\frac{\left(x+d_G(v_n)+1\right)\left(x+d_H(u_1)+1\right)}{\left(x+d_G(v_n)\right)\left(x+d_H(u_1)\right)}\times NK\left(G,x\right)NK\left(H,x\right).$$

■

The results for splice can be in a straightforward way generalized to more than two operands (see [14]). If we have graphs $G_1,\ldots,G_k$ and $v_i\in V(G_i)$ for each $i = 1, \ldots, k$, then their splice in vertices v_i is obtained by identifying all k vertices v_i.

Corollary 2.5. Consider the graphs $G_1,\ldots,G_k$ and $v_i\in V(G_i)$ for each $i = 1, \ldots, k$, then

$$NK(G_1\bullet G_2\bullet\cdots\bullet G_k,x)=\prod_{i=1}^{k}NK(G_i,x)\times\frac{\left(x+\sum_{i=1}^{k}d_{G_i}(v_i)\right)}{\prod_{i=1}^{k}\left(x+d_{G_i}(v_i)\right)}.$$

If we have k copies of the same graph G and splice them at the same vertex v, we obtain $G^{\bullet k}$, the k-th *splice-power* of G. The above result then simplifies to

$$NK(G^{\bullet k})=\frac{k\left[NK(G)\right]^k}{d_G(v)^{k-1}}.$$

The links of more than two graphs is known as the *chain (or bridge)* graphs. Let G_i, $1\le i\le k$, be some graphs and $v_i\in V(G_i)$. A chain graph denoted by $G=G(G_1,\ldots,G_k,v_1,\ldots,v_k)$ is obtained from the union of the graphs G_i, $i = 1, \ldots, k$, by adding the edges v_iv_{i+1}, $1\le i\le k-1$.

Then $|V(G)|=\sum_{i=1}^{k}|V(G_i)|$ and $|E(G)|=(k-1)+\sum_{i=1}^{k}|E(G_i)|$.

Obviously $G(G_1,G_2,v_1,v_2)\cong(G_1\sim G_2)(v_1,v_2)$.

It is worth noting that the above specified class of chain graphs embraces, as special cases, all trees (among which are the molecular graphs of alkanes) and all unicyclic graphs (among which are the molecular graphs of monocycloalkanes). Also the molecular graphs of many polymers and dendrimers are chain graphs.

Further, when all G_i are equal to G, the chain graph becomes a rooted product of the path on k vertices and G, and if, in addition, all v_i are equal, we have the crown of P_k and G.

Let $G=G(G_1,\ldots,G_k,v_1,\ldots,v_k)$ is a chain graph. Then obviously the following holds.

$$d_G(u)=\begin{cases}d_{G_i}(u) & if\ u\in V(G_i)\ and\ u\ne v_i\\ d_{G_i}(v_i)+1 & if\ u=v_i, i=1,k\\ d_{G_i}(v_i)+2 & if\ u=v_i, 2\le i\le k-1\end{cases}.$$

Theorem 2.6. For the chain graph $G=G(G_1,\ldots,G_k,v_1,\ldots,v_k)$, we have

$$NK(G(G_1,\ldots,G_k,v_1,\ldots,v_k),x)=(x+d_{G_1}(v_1)+1)$$
$$\times(x+d_{G_k}(v_k)+1)\prod_{i=2}^{k-1}(x+d_{G_1}(v_i)+2)$$
$$\times\prod_{i=1}^{n}NK^{(V(G_i)-v_i)}(G_i,x).$$

Proof. By definition of a chain graph, the degree of each vertex v in a chain graph, except v_i's, is as the same as in its underlying graph. Also, the degrees of v_1 and v_2 are added by 1 and the degrees of $v_2, \ldots, v_{n-1}$ are added by 2, which these yield to the result. ■

Now, we obtain the Narumi-Katayama polynomial of some graph operations whose Narumi-Katayama index has computed in [14]. We, also, generalize some of the proof techniques in [14]. The *join* of two graphs G_1 and G_2 is obtained by taking their union and adding all possible edges between $V(G_1)$ and $V(G_2)$. We denote it by $G_1\nabla G_2$. When one of the graphs is K_1, the join of K_1 and G is called the *suspension* of G.

Theorem 2.7. Let G be a graph. Then

$$NK(K_1\nabla G,x)=\left(x+n\right)\sum_{U\subseteq V(G)}NK^{(U)}(G,x).$$

Proof. The degree of a vertex of G in its suspension increases by one, while the degree of the vertex of K_1 is equal to $|V(G)| = n$. Hence the Narumi-Katayama polynomial of $K_1\nabla G$ is given by

$$NK(K_1\nabla G,x)=\left(x+n\right)\prod_{i=1}^{n}(x+d_G(v_i)+1).$$

The product on the right-hand side of the above formula can be expressed in terms of truncated Narumi-Katayama polynomial with respects to all subsets of $V(G)$. The result follows by expanding the product into a sum of 2^n terms and noting that the products of the sum of x and degrees of each of 2^n subsets of $V(G)$ appear exactly once in the sum. ■

The above result can be straightforwardly generalized to the case when one of the components of a join is the set of m isolated vertices, i.e., the complement $\bar{K}_m$ of the complete graph K_m.

Theorem 2.8. For a graph G, we have

$$NK(\bar{K}_m\nabla G,x)=\left(x+n\right)^m \times\left(\sum_{U\subseteq V(G)}NK^{(U)}(G,x)m^{n-|U|}\right).$$

A closer look on the above formula should reveal that all effects of the independence of vertices of $\bar{K}_m$ are concentrated in the n^m term. Hence we can conclude the following result.

Theorem 2.9. Let G_1 and G_2 be two graphs with n_1 and n_2 vertices, respectively. Then

$$NK(G_1\nabla G_2,x)=\left(\sum_{U_1\subseteq V(G_1)}NK^{(U_1)}(G_1,x)n_2^{n_1-|U_1|}\right)\times\left(\sum_{U_2\subseteq V(G_2)}NK^{(U_2)}(G_2,x)n_1^{n_2-|U_2|}\right).$$

Proof. The contribution of vertices of one component in the join of two graphs depends only on the number of vertices in the other component, and not on its internal structure. From this observation we can deduce the formula for the general case as below:

$$NK(G_1\nabla G_2,x)=\left(\prod_{i=1}^{n_1}(x+d_{G_1}(v_i)+n_2)\right)\times\left(\prod_{i=1}^{n_2}(x+d_{G_2}(v_i)+n_1)\right)$$
$$=\left(\sum_{U_1\subseteq V(G_1)}NK^{(U_1)}(G_1,x)n_2^{n_1-|U_1|}\right)\times\left(\sum_{U_2\subseteq V(G_2)}NK^{(U_2)}(G_2,x)n_1^{n_2-|U_2|}\right).$$

■

The *corona* of two graphs G and H is the graph obtained by taking $|V(G)|$ copies of H and connecting each vertex in the i-th copy of H to the vertex v_i of G. It is usually denoted by $G\circ H$.

Theorem 2.10. Let G and H be two graphs with n and m vertices, respectively. Then

$$NK(G\circ H,x)=\left(\sum_{U\subseteq V(G)}NK^{(U)}(G,x)m^{n-|U|}\right)\times\left(\sum_{W\subseteq V(H)}NK^{(W)}(H,x)\right)^n.$$

Proof. A corona is a collection of n suspensions of H on a scaffold provided by G. The degree of each vertex in corona of two graphs is

$$d_{G\circ H}(v)=\begin{cases} d_G(v)+m & v\in V(G) \\ d_H(v)+1 & v\in V(H)\end{cases}$$

So we have

$$NK(G\circ H,x)=\prod_{v\in V(G\circ H)}(x+d_{G\circ H}(v))$$

$$=\left(\prod_{v\in V(G)}\left(x+d_G(v)+m\right)\right)$$

$$\times\left(\prod_{u\in V(H)}\left(x+d_H(u)+1\right)\right)$$

$$=\left(\sum_{U\subseteq V(G)}NK^{(U)}(G,x)m^{n-|U|}\right)$$

$$\times\left(\sum_{W\subseteq V(H)}NK^{(W)}(H,x)\right)^n.$$

■

The *composition* of two graphs G and H is the graph with vertex set $V(G) \times V(H)$, and the vertex $u = (u_1, v_1)$ is adjacent to the vertex $v = (u_2, v_2)$ whenever either $u_1u_2 \in E(G)$ or $u_1 = u_2$ and $v_1v_2 \in E(H)$. This graph operation is denoted by $G[H]$. The composition of two graphs is also known as graph substitution, a name that bears witness to the fact that $G[H]$ can be obtained from G by substituting a copy of H, labeled H_w, for every vertex w in $V(G)$ and then joining all vertices of H_w with all vertices of $H_{w'}$ if and only if $ww' \in E(G)$, and there are no edges between vertices in H_u and $H_{u'}$ otherwise. Now by the above approach, one can see the Narumi-Katayama polynomial of the composition of two graphs as follow:

Theorem 2.11. Let G and H be two graphs with n and m vertices, respectively. Then

$$NK(G[H],x)=\prod_{u\in V(G)}\sum_{U\subseteq V(H)}NK^{(U)}(H,x)\left(nd_G(u)\right)^{m-|U|}.$$

Proof. The degree of the vertex (u,v) in $G[H]$ is $d_{G[H]}(u,v) = d_H(v) + nd_G(u)$, where n is the number of vertices of G. So we have

$$NK(G\circ H,x)=\prod_{(u,v)\in V(G\circ H)}\left(x+d_{G\circ H}\left((u,v)\right)\right)$$

$$=\prod_{u\in V(G)}\prod_{v\in V(H)}\left(x+d_H(v)+nd_G(u)\right)$$

$$=\prod_{u\in V(G)}\sum_{U\subseteq V(H)}NK^{(U)}(H,x)\left(nd_G(u)\right)^{m-|U|}.$$

■

The *Cartesian product* $G_1 \times G_2$ of graphs G_1 and G_2 is a graph such that $V(G_1 \times G_2) = V(G_1) \times V(G_2)$, and any two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G_1 \times G_2$ if and only if either $u_1 = u_2$ and v_1 is adjacent with v_2, or $v_1 = v_2$ and u_1 is adjacent with u_2. Now, we complete this section by computing the Narumi-Katayama polynomial of Cartesian product of two graphs.

Theorem 2.12. Let G_1 and G_2 be two graphs with n and m vertices, respectively. Then

$$NK(G_1\times G_2,x)=\frac{1}{2}\Big(\prod_{u\in V(G_1)}\sum_{U\subseteq V(G_2)}NK^{(U)}(G_2,x)\left(d_{G_1}(u)\right)^{m-|U|}$$

$$+\prod_{v\in V(G_2)}\sum_{W\subseteq V(G_1)}NK^{(W)}(G_1,x)\left(d_{G_2}(v)\right)^{n-|W|}\Big).$$

Proof. One can see that the degree of the vertex (u,v) in Cartesian product of two graphs G_1 and G_2 is $d_{G_1\times G_2}(u,v)= d_{G_1}(u)+ d_{G_2}(v)$. We can write

$$NK(G_1\times G_2,x)=\prod_{(u,v)\in V(G_1\times G_2)}\left(x+d_{G_1\times G_2}\left((u,v)\right)\right)$$

$$=\prod_{u\in V(G_1)}\prod_{v\in V(G_2)}\left(x+d_{G_2}(v)+d_{G_1}(u)\right)$$

$$=\prod_{u\in V(G_1)}\sum_{U\subseteq V(G_2)}NK^{(U)}(G_2,x)\left(d_{G_1}(u)\right)^{m-|U|}.$$

Similarly, we have

$$NK(G_1\times G_2,x)=\prod_{(u,v)\in V(G_1\times G_2)}\left(x+d_{G_1\times G_2}\left((u,v)\right)\right)$$

$$=\prod_{v\in V(G_2)}\prod_{u\in V(G_1)}\left(x+d_{G_1}(u)+d_{G_2}(v)\right)$$

$$=\prod_{v\in V(G_2)}\sum_{W\subseteq V(G_1)}NK^{(W)}(G_1,x)\left(d_{G_2}(v)\right)^{n-|W|}.$$

So to preserve the symmetric of the formula for the Narumi-Katayama polynomial of Cartesian product of two graphs, we have the assertion. ■

2. NARUMI - KATAYAMA POLYNOMIAL OF SOME NANO STRUCTURE

In this section we use some of our results in Section 2 to compute the Narumi-Katayama polynomial of some nanostructures. To do this, we start with a class of nanostar dendrimers such that they have an inductive structure and are denoted by $D'_1[n]$, (see [13]). In computer science, a binary tree is a tree data structure in which each node has at most two child nodes, usually distinguished as left and right. Nodes with children are parent nodes, and child nodes may contain references to their parents. Outside the tree, there is often a reference to the root node (the ancestor of all nodes), if it exists. Any node in the data structure can be reached by starting at root node and repeatedly following references to either the left or right child. The graph $D'_1[3]$ is shown in Figure 1.

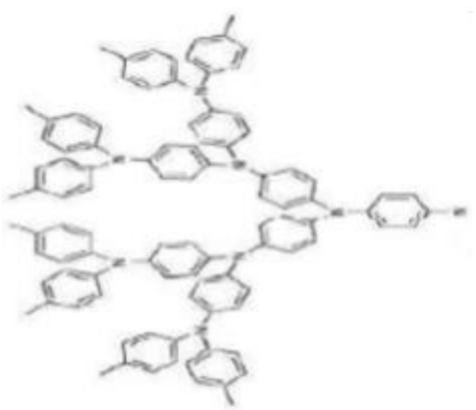

Figure 1. *The graph* $D'_1[3]$.

Theorem 3.1. For the graph $D'_1[n]$ we have

$$NK(D'_1[n],x)=(x+1)^{2^n+1}(x+2)^{2^{n+3}-4}(x+3)^{5(2^n-2)+7}$$

Proof. To prove the theorem, we apply induction on n. If $n = 1$, then

$$NK(D'_1[1],x)=(x+1)^3(x+2)^{12}(x+3)^7$$

and the assertion holds. Therefore, let $n > 1$. By construction of nanostar dendrimer $D'_1[n]$, one can see that $D'_1[n]$ is obtained from the splice of $D'_1[n]$ and $2n$-1 copies of the graph H in Figure 2.

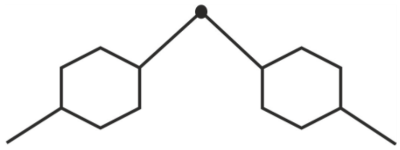

Figure 2. *The graph H.*

Thus, by Theorem 2.4, we have

$$NK(D'_1[n],x)=\frac{NK(D'_1[n-1],x)}{(x+1)^{2^{n-1}}}\times(x+3)^{5\times2^{n-1}}(x+2)^{2^{n+2}}$$

$$=\frac{(x+1)^{2^{n-1}+1}(x+2)^{2^{n+2}-4}(x+3)^{5(2^{n-1}-2)+7}}{(x+1)^{2^{n-1}}}\times(x+1)^{2^n}(x+3)^{5\times2^{n-1}}(x+2)^{2^{n+2}}$$

$$=(x+1)^{2^n+1}(x+2)^{2^{n+3}-4}(x+3)^{5(2^n-2)+7}$$

and the proof is completed. ■

Now, we apply our results to compute the Narumi-Katayama polynomial of a nanostar dendrimer. We consider the first kind of nanostar dendrimer which has grown n steps denoted $D_3[n]$, [13]. The nanostar dendrimer $D_3[n]$ is depicted in Figure 3.

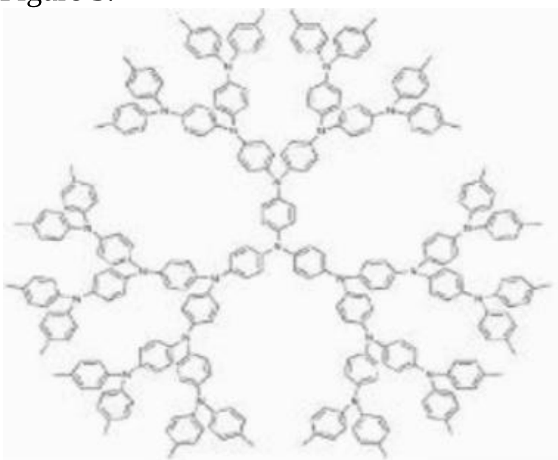

Figure 3. *The first kind of nanostar dendrimer of generation 1-3 has grown 3 stages.*

First we construct $D_3[n]$ from $D'_1[n]$. Consider three copies of $D'_1[n]$ with roots O, P and Q. The nanostar dendrimer $D_3[n]$ is obtained by identifying the vertex O of one copy of $D'_1[n]$ with two vertices P and

Q of two another copies of $D'_1[n]$. We have the following theorem:

Theorem 3.2. Consider the graph nanostar dendrimer $D_3[n]$. We have

$$NK(D_3[n],x) = (x+1)^{3\times 2^n}(x+2)^{3(2^{n+3}-4)} \times (x+3)^{15(2^n-2)+22}$$

Proof. By definition of $D_3[n]$, one can see that if we consider the splice of two copies of $D'_1[n]$ by vertices O and P, then the splice of the resulting graph with another copy of $D'_1[n]$ by vertex Q will be the graph $D_3[n]$. So, using the Theorem 2.4 and Theorem 3.1, we have

$$NK\left(\left(D'_1[n]\bullet D'_1[n]\right)(O,P),x\right)=\frac{(x+2)}{(x+1)^2}(x+1)^{2^n+1}$$
$$\times(x+2)^{2^{n+3}-4}(x+3)^{5(2^n-2)+7}(x+1)^{2^n+1}(x+2)^{2^{n+3}-4}$$
$$\times(x+3)^{5(2^n-2)+7}$$
$$=(x+1)^{2^{n+1}}(x+2)^{2^{n+4}-7}(x+3)^{10(2^n-2)+14}$$

Therefore

$$NK(D_3[n],x)=\frac{(x+3)}{(x+1)(x+2)}$$
$$\times(x+1)^{2^n+1}(x+2)^{2^{n+3}-4}(x+3)^{5(2^n-2)+7}$$
$$\times(x+1)^{2^{n+1}}(x+2)^{2^{n+4}-7}(x+3)^{10(2^n-2)+14}$$
$$=(x+1)^{3\times 2^n}(x+2)^{3(2^{n+3}-4)}(x+3)^{15(2^n-2)+22}$$

which this complete the proof. ■

Now, we provide another example to show the application of our result about the link of two graphs. In Figure 4, the link of two fullerene graph is showed [17].

Theorem 3.3. Consider the link of tow fullerene graph C_{20} in Figure 4. We have

$$NK(C_{20}\sim C_{20},x)=(x+3)^{38}(x+4)^2$$

Proof. Since the fullerene graph C_{20} is a *3*-regular graph with 20 vertices,

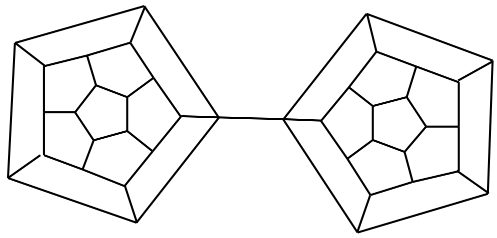

Figure 4. *Link Graph* $C_{20}\sim C_{20}$.

$NK(C_{20},x) = (x+3)^{20}$. Now, by Theorem 2.4 we have

$$NK(C_{20}\sim C_{20},x)=\frac{(x+4)^2}{(x+3)^2}\left(NK(C_{20},x)\right)^2$$
$$=\frac{(x+4)^2}{(x+3)^2}(x+3)^{20}(x+3)^{20}$$
$$=(x+3)^{38}(x+4)^2.$$ ■

Now, we are ready to compute the Narumi-Katayama polynomial of the chain of fullerene graphs in Figure 5. This graph has an inductive structure which we denote it by $C_{20}\overset{n}{\sim}C_{20}$, where n is the number of copies of the fullerene graph C_{20}, see [17].

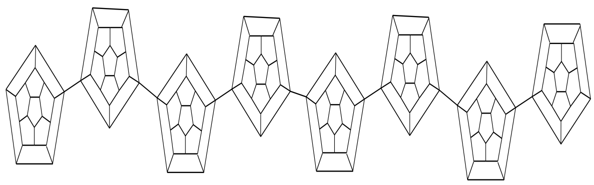

Figure 5. *The chain of fullerene graphs* C_{20}.

Theorem 3.4. The Narumi-Katayama polynomial of the graph $C_{20}\overset{n}{\sim}C_{20}$ is as follows

$$NK\left(C_{20}\overset{n}{\sim}C_{20},x\right)=(x+3)^{18n+2}(x+4)^{2(n-1)}$$

Proof. We prove this theorem by induction on n. Using Theorem 3.3, the assertion holds for $n=2$. So, suppose that $n > 2$. Since $C_{20}\overset{n}{\sim}C_{20}=C_{20}\overset{n-1}{\sim}C_{20}\sim C_{20}$, by Theorem 2.4 we have

$$NK\left(C_{20}\overset{n}{\sim}C_{20},x\right)=NK\left(C_{20}\overset{n-1}{\sim}C_{20}\sim C_{20},x\right)$$
$$=\frac{(x+4)^2}{(x+3)^2}NK\left(C_{20}\overset{n-1}{\sim}C_{20},x\right)NK\left(C_{20},x\right)$$
$$=\frac{(x+4)^2}{(x+3)^2}(x+3)^{18n-16}(x+4)^{2(n-2)}(x+3)^{20}$$
$$=(x+3)^{18n+2}(x+4)^{2(n-1)}$$
and the proof is completed. ■

3. CONCLUSION

Until now, for more investigation of topological indices, many polynomials are associated to different topological indices, as instance we refer the readers to [6, 13, 16]. In this paper, we continue this process and we introduce the concept of Narumi-Katayama polynomial of a graph. This is for the first time which a polynomial is associated to a multiplicative topological index. Then, some graph theoretical properties of this polynomial is computed. For example, we compute this polynomial for the sun graph which its Nk index was computed in [18]. Also, we obtain a relationship between the coefficient of the variable x in this polynomial and the truncated Narumi-Katayama index. Next, the Narumi-Katayama polynomial of some graph operations such as splice and link of two graphs are computed. Also, we have investigated the behavior of the Narumi-Katayama polynomial under several binary operations resulting in composite graphs. In most cases we obtained formulas containing sums of truncated Narumi-Katayama polynomial with respect to all subsets of the vertex sets of considered graphs. Furthermore, there are some interesting operations we have not considered here. Finally, our results on these graph operations are applied to compute the Narumi-Katayama polynomial of a class of nanostar dendrimers and the link of fullerene graphs C_{20}. By this methods and techniques, Narumi-Katayama polynomial of many families of nanostar dendrimers and the link of other fullerene graphs which are not mentioned in here, can be computed. Notice that, since the Narumi-Katayama index of a graph can be obtained by Narumi-Katayama polynomial of that graph, so our results are applicable for Narumi-Katayama index of graphs.

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