Int. J. Nanosci. Nanotechnol., Vol. 11, No. 3, Sept. 2015, pp. 159-166

Analytical Solution for the Forced Vibrations of a Nano-Resonator with Cubic Nonlinearities Using Homotopy Analysis Method

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(Received: 25 Dec 2014 and Accepted: 10 Sep. 2015)

Abstract
Many of nonlinear systems in the field of engineering such as nano-resonator and atomic force microscope can be modeled based on Duffing equation. Analytical frequency response of this system helps us analyze different interesting nonlinear behaviors appearing in its response due to its rich dynamics. In this paper, the general form of Duffing equation with cubic nonlinearity as well as parametric excitations is considered and its frequency response is derived utilizing Homotopy Analysis Method (HAM) for the first time. Although time response of different Duffing systems has been analyzed using HAM, derivation of its frequency response equation by applying this powerful method has not been presented. The main advantage of proposed simple closed-form solution is that it is not restricted to weakly nonlinear systems in contrast with perturbation methods. Because of numerous applications of Micro-electro-mechanical resonator and its rich and nonlinear dynamics, it is considered as a case study in this paper and the obtained analytical equation is applied to find its frequency response. The validation of analytical method is verified by comparing the results with numerical simulations. It is also shown that proposed closed-form equation for nano-resonator frequency response can capture both hardening and softening behavior of nano-resonator as well as jump phenomenon. The results of this paper can be useful in analysis of different engineering systems modeled by general Duffing equation.
Keywords*: Duffing equation, Frequency response, Homotopy analysis method, Nano-resonators.*

1. INRODUCTION

Nonlinear vibrating models have been widely used in different fields of engineering and are of significant importance in mechanical dynamics for the comprehensive analysis and accurate prediction of response of systems. The Duffing equation characterizes the class of vibrating systems with cubic nonlinearities. Due to its rich dynamics, it shows a wide variety of nonlinear phenomena such as bi-stability, chaotic motion and jump phenomenon in which the steady state response of system jumps from one stable solution to another stable one. The analysis of this low-order nonlinear system can be helpful in the development of reduced order models of complex mechanical systems ranging from micro-scales to macro-scales (Schuster, 2008; Shaw and Balachandran, 2008)

Frequency response of a nonlinear system can reveal its different nonlinear behaviors such as hardening and softening behaviors, bi-stability and jump phenomenon(Miandoab et al., 2014b). There are many approaches for approximating frequency response of Duffing system. The most common and most widely used methods of them are the perturbation methods such as Multiple scales (Caruntu and Luo, 2014), averaging (Fahsi and Belhaq, 2009) and lindested-poincare methods(Chen and Cheung, 1996). These methods involve the expansion of a solution for a differential equation in the series of powers of the small parameter and the main drawback of them is that they can only be used for weakly nonlinear systems.

One of the other powerful methods to approximate the frequency response of nonlinear systems is Volterra series approach (Worden and Manson, 1998). The analytical approximations based on the Volterra series are in good agreement with numerical simulations but because of two main reasons it is less impressive. First, it is difficult to estimate how many terms of the series are needed to guarantee the series converge and the second is the complexity and growing of the calculation by increasing the term order (Lang et al., 2007; Worden and Tomlinson, 2010).

Lio (Liao, 2003) developed the Homotopy Analysis Method (HAM) which is a powerful method to solve highly nonlinear equations. Unlike other methods, this method does not depend on any assumptions of small parameter and HAM can guarantee convergence of the series solution by introducing a convergence-control parameter (Liao, 2012). Although HAM has been widely implemented in the literature for analysis of time response of various highly nonlinear problems (Abbasbandy, 2006; Van Gorder and Vajravelu, 2008; Tajaddodianfar et al., 2015), derivation of nonlinear frequency response of system using this method has not been presented in the literature until now. Frequency response of nonlinear system reveals its rich dynamics such as hardening, softening, jump and bi-stability. Since the equation of motion of many mechanical systems ranging from automotive belt-pulley system (Michon et al., 2008) to Atomic force microscope (Pishkenari et al., 2008) and MEMS/NEMS systems (Miandoab et al., 2014c) leads to parametrically actuated Duffing equation, this model is considered in this study and its analytical frequency response is approximated using Homotopy Analysis Method.

MEMS resonators are one of the most commonly used components for various applications from accelerometers to communication and signal processing devices (Tocchio et al., 2012; Hajjam and Pourkamali, 2012). Typically these systems include one movable electrode excited by DC and AC voltages applied on one or both sides. Different nonlinearity sources such as mid-plane stretching, squeezing film damping and nonlinear coupling between electrostatic force and resonator displacement lead to nonlinear behavior of resonators such as frequency response curve bending, jump phenomenon and chaotic motion (Ghayesh et al., 2013). Deriving the analytical frequency response of resonators reveals different nonlinear behaviors of these systems. For example, Maani Miandoab et al. have recently showed that analytical frequency response of multi-well potential systems can be utilized to accurate prediction of chaos in these systems (Miandoab et al., 2014a).

MEMS resonator under combined AC and DC actuations is considered as the case study and proposed closed form analytical frequency response is used to obtain its frequency response. Comparing the results with numerical simulations illustrates best agreement. In section 2, we introduce the implementation of Homotopy Analysis Method to parametric Duffing equation. Section 3 contains the Derivation of MEMS resonator equation and comparison of the analytical and numerical results. The paper ends with conclusions drawn from our research in section 5.

2. FREQUENCY RESPONSE BY THE HAM

In this section, we aim to derive analytical solution for the response amplitude of forced oscillatory behavior of a system having cubic nonlinearities:

$$\frac{d^2u}{d\tau^2}+\mu\frac{du}{d\tau}+\lambda^2u+\xi u^3= R(\mathrm{r}_0+\mathrm{r}_1\mathrm{u}+\mathrm{r}_2\mathrm{u}^2+\mathrm{r}_3\mathrm{u}^3)\cos\ \Omega\tau \tag{1}$$

Where λ denotes the linear natural frequency of the system. Liao has already proposed and developed the fundamentals of the Homotopy Analysis Method (HAM) (Liao, 2003; Liao, 2009). Here, we implement this technique to derive approximate solution for the response amplitude of a forced oscillator governed by Eq. 1. Regarding the harmonic excitation of the system, we suppose that the response of the oscillator is possible to be expressed in the form of a convergent series:

$$u(\tau)=\sum_{k=1}^{\infty}U_k e^{ik\omega\tau}+\bar{U}_k e^{-ik\omega\tau}, \tag{2}$$

Where U_k and $\bar{U}_k$ represent two complex conjugate constants. This assumption is known as the "rule of solution expression" in the homotopy analysis method [1]. According to the well-known HAM procedure, we propose a parameter $p\in[0,1]$ and a variation $\varphi(\tau,p):\mathbb{R}^+\times[0,1]\rightarrow\mathbb{R}$ such that $\varphi(\tau,\mathrm{p})$ matures from a primary guess $\varphi(\tau,0)=u_0(\tau)$ to the exact solution $\varphi(\tau,1)=u(\tau)$, as p continuously varies from 0 to 1. Regarding hypothesized series solution 2, we can select an appropriate initial guess:

$$u_0(\tau)=\mathrm{U}e^{i\omega\tau}+\bar{U}e^{-i\omega\tau} \tag{3}$$

Then, we can define a differential operator called the "auxiliary linear operator" in the HAM approach:

$$\mathrm{L}\,[\varphi(\tau,\mathrm{p})]=\frac{\partial^2\varphi(\tau,\mathrm{p})}{\partial\tau^2}+\omega^2\varphi(\tau,\mathrm{p}) \tag{4}$$

which satisfies the condition below:

$$\mathrm{L}\,[\mathrm{U}_1e^{i\omega\tau}+\bar{\mathrm{U}}_1e^{-i\omega\tau}]=0 \tag{5}$$

Then, using Eq. 1, the following nonlinear operator is proposed:

$$\mathrm{N}(\varphi(\tau,\mathrm{p}))=\frac{\partial^2\varphi}{\partial\tau^2}+\mu\frac{\partial\varphi}{\partial\tau}+\lambda^2\varphi(\tau,\mathrm{p})+\xi\varphi(\tau,\mathrm{p})^3 -R(\mathrm{r}_0+\mathrm{r}_1\mathrm{u}+\mathrm{r}_2\mathrm{u}^2+\mathrm{r}_3\mathrm{u}^3)\cos((\omega+\sigma)\tau) \tag{6}$$

The zeroth-order deformation equation, as a basic equation in the HAM procedure, is defined using a further auxiliary parameter known as the convergence-control parameter, c_0 :

$$(1-p)\mathrm{L}\,[\varphi(\tau,\mathrm{p})-u_0(\tau)]=c_0p\,\mathrm{N}\,(\varphi(\tau,\mathrm{p})) \tag{2}$$

We suppose a Maclaurin series in p exists such that it converges to the variation $\varphi(\tau,\mathrm{p})$ using the deformation derivatives $u_k(\tau)$ given as below:

$$u_k(\tau)=\frac{1}{k!}\frac{\partial^k\varphi(\tau,\mathrm{p})}{\partial p^k}\bigg|_{p=0} \tag{8}$$

$$\varphi(\tau,\mathrm{p})=\sum_{k=0}^{\infty}u_k(\tau)\,\mathrm{p}^k \tag{9}$$

However, the deformation derivatives, and consequently the series solution for

$\varphi(\tau,\mathrm{p})$, are found by solving the following k-th order deformation equations [1]:

$$\mathrm{L}\,[u_k(\tau)-\chi_k u_{k-1}(\tau)] = c_0 R_k(\tau), \qquad (10)$$

$$R_k(\tau) = \frac{1}{(\mathrm{k}-1)!}\frac{\partial^{k-1}}{\partial p^{k-1}}\,\mathrm{N}\,[\varphi(\tau,\mathrm{p})]\Bigg|_{p=0}, \qquad (11)$$

$$\chi_k = \begin{cases} 0 & k \le 1 \\ 1 & k > 1 \end{cases}, \qquad (12)$$

However, using initial guess of Eq 3 together with Eqs. 10, 11 and 12, we are left with the following equation for $u_1(\tau)$:

$$\frac{\partial^2 u_1(\tau)}{\partial \tau^2} + \omega^2 u_1(\tau) = c_0\,\mathrm{N}\,[u_0(\tau)] \qquad (13)$$

In order to prevent unbounded terms in the solution of the above equation, secular terms should be set to zero which yields:

$$-\left(U\bar{U}r_2 - \frac{r_0}{2}\right)Re^{i\sigma\tau} - \frac{1}{2}r_2U^2\,\mathrm{Re}^{-i\sigma\tau} + \qquad (14)$$
$$3\xi U^2\bar{U} + U\mu\omega i + U\ \ \lambda^2 - \omega^2\ \ = 0$$

Supposing a and b to be real parameters representing amplitude and phase shift of the response, $U = \frac{1}{2}ae^{ib}$ and $\bar{U} = \frac{1}{2}ae^{-ib}$ are introduced in Eq. 14. Multiplying both sides by e^{-ib}, equating the real and imaginary parts of the resulting equation to zero, and pursuing the simple trigonometric calculations, we are left with the following equation for the frequency response of the oscillator:

$$\left(\frac{3\xi a^2 + 4\ \ \lambda^2 - \omega^2}{4r_0 + 3r_2a^2}\right)^2 + \left(\frac{4\mu\omega}{4r_0 + r_2a^2}\right)^2 = \left(\frac{R}{a}\right)^2 \qquad (15)$$

Regarding the Eq. 1, all parameters are known in Eq. 15 with $\omega = \Omega$. Thus, one can solve Eq. 15 for the unknown a describing the amplitude of the vibrations. This result is validated through a case study in the next section.

3. CASE STUDY: MEMS RESONATOR

Figure1 shows a micro-electro-mechanical-resonator under electrostatic actuation.

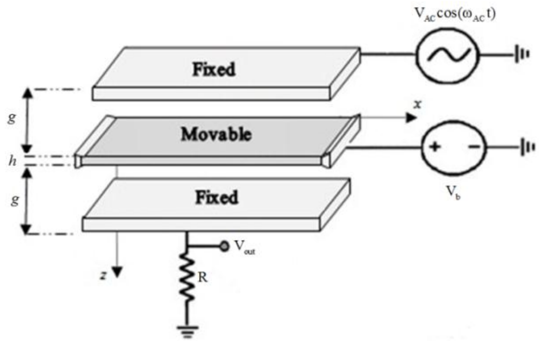

Figure1. *Schematic of an electrostatically actuated micro resonator.*

As can be seen in Figure1, the resonator is composed of two fixed and one movable electrode. A DC load is applied to the middle movable electrode while one of the fixed electrodes is under harmonic AC actuation. This combination of loadings makes an oscillatory system with a bias DC and harmonic AC voltage. Different nonlinearities in this system such as electrostatic force and mid-plane stretching may lead to interesting nonlinear behaviors such as frequency response bending, jump phenomenon and chaotic vibration which affect the performance of this device and is important in design, analysis and fabrication of nano-resonators. Closed form frequency response of these systems is a fast and useful method to analyze these behaviors in comparison with numerical methods which are time consuming. Thus, in this section, HAM will be used to analyze the frequency response of this system and its validation will be verified by comparing the results with numerical ones. By considering the movable electrode as double clamped micro-beam, the governing equation of the micro-resonator can be written as follows

(Ouakad and Younis, 2012; Younis and Nayfeh, 2003), where we assumed Euler-Bernoulli beam model.

$$EI\frac{\partial^4 w}{\partial x^4}+\rho A\frac{\partial^2 w}{\partial t^2}+C\frac{\partial w}{\partial t}=\left[N+\alpha_1\int_0^1(\frac{\partial w}{\partial x})^2dx\right]\frac{\partial^2 w}{\partial x^2}$$
$$+\varepsilon bV_b^2(\frac{1}{(g-w)^2}-\frac{1}{(g+w)^2})+\frac{2\varepsilon bV_bV_{Ac}\cos(\omega_{ac}t)}{(g-w)^2} \quad (16)$$

In the above equation, the parameters E, A, ρ, I, ε, N, g and w are respectively the micro-beam Young's modulus, its cross-sectional area, material density, moment of inertia, air dielectric constant, the applied axial force, the gap distance between the two electrodes and the beam deflection which is a function of the axial position, x , as well as time, t .

To start tackling the resolution of Eq. 16, it is more convenient to use the following non-dimensional parameters

$$\hat{x}=\frac{x}{l};\ \hat{w}=\frac{w}{g};\ \hat{t}=\frac{t}{T} \quad (17)$$

Substituting 17 in 16 results in:

$$T(\hat{w})=\frac{\partial^4\hat{w}}{\partial\hat{x}^4}+\frac{\partial^2\hat{w}}{\partial\hat{t}^2}+c\frac{\partial\hat{w}}{\partial\hat{t}}-(\hat{N}+\alpha\int_0^1(\frac{\partial\hat{w}}{\partial\hat{x}})^2d\hat{x})\frac{\partial^2\hat{w}}{\partial\hat{x}^2}$$
$$-\lambda(\frac{1}{(1-\hat{w})^2}-\frac{1}{(1+\hat{w})^2})-\frac{\kappa\cos(\Omega_{ac}\hat{t})}{(1-\hat{w})^2}=0$$
(18)

Where:

$$c=\frac{Cl^4}{EIT},\quad \hat{N}=\frac{l^2}{EI}N,\quad \alpha=\frac{EAg^2}{2EI},\quad \lambda=\frac{\varepsilon bl^4V_b^2}{2EIg^3},$$
$$\kappa=2\lambda\frac{V_{AC}}{V_b},\quad T=\sqrt{\frac{\rho bhl^4}{EI}},\ \Omega_{ac}=T\omega_{ac}$$

(19)

The boundary conditions corresponding to double clamped micro-beam are given as follows:

$$\hat{w}(0,t)=\hat{w}(1,t)=0; \quad (20)$$

$$\frac{\partial\hat{w}(0,t)}{\partial\hat{x}}=\frac{\partial\hat{w}(1,t)}{\partial\hat{x}}=0$$

Based on the separation of variables technique, the solution of Eq. 3 can be approximated as in the following form: (6)

$$\hat{w}(\hat{x},\hat{t})\simeq\varphi(\hat{x})u(\hat{t}) \quad (21)$$

Where, $\varphi(\hat{x})$ is the first mode shape of the double clamped beam. The governing differential equation of $u(\hat{t})$ is derived by substituting Eq. 6 into Eq. 3 and using the Galerkin's decomposition (Ouakad and Younis, 2014; Ouakad, 2013).

$$\int_0^1\varphi(\hat{x})T(\hat{w})d\hat{x}=0 \quad \Rightarrow \quad (22)$$
$$\ddot{u}+\alpha_1u+\alpha_3u^3+c\dot{u}=F_e$$

Where:

$$\alpha_1=\int_0^1(\varphi(\hat{x})\frac{\mathrm{d}^4\varphi(\hat{x})}{\mathrm{d}\hat{x}^4}-\hat{N}\varphi(\hat{x})\frac{\mathrm{d}^2\varphi(\hat{x})}{\mathrm{d}\hat{x}^2})\mathrm{d}\hat{x},$$
$$\alpha_3=-\alpha(\int_0^1(\frac{\mathrm{d}\varphi(\hat{x})}{\mathrm{d}\hat{x}})^2\mathrm{d}\hat{x})\int_0^1\varphi(\hat{x})\frac{\mathrm{d}^2\varphi(\hat{x})}{\mathrm{d}\hat{x}^2}d\hat{x} \quad (23)$$

And where:

$$F_e=\int_0^1\varphi(\hat{x})(\frac{\lambda-\kappa\cos(\Omega_{ac}\hat{t})}{(1-u\varphi(\hat{x}))^2}-\frac{\lambda}{(1+u\varphi(\hat{x}))^2})\mathrm{d}\hat{x} \quad (24)$$

To find the lumped force applied on the micro-resonator, in (Miandoab et al., 2014b) it was supposed that the nonlinear integral function is possible to be approximated by a function having known structure:

$$\int_0^1(\frac{\varphi(\hat{x})}{(1-u\varphi(\hat{x}))^2}d\hat{x}=\frac{\eta_0}{(1-\eta_1u)^{\eta_2}} \quad (25)$$

The unknown parameters were found using Genetic Algorithm as: (7)

$$\eta_0 = 0.8309 \tag{26}$$
$$\eta_1 = 1.5837$$
$$\eta_2 = 1.5196.$$

By introducing $\tau = \sqrt{\alpha_1}\,\hat{t}$ and implementing the obtained approximations, Eq. 22 reduces to:

$$u'' + u + \beta u^3 + \mu u' = 0.8309\left(\frac{\gamma - A\cos(\Omega_0 \tau)}{(1-1.5837u)^{1.5196}}\right) - \frac{\gamma}{(1+1.5837u)^{1.5196}}) \tag{27}$$

With the parameters described below:

$$\beta = \frac{\alpha_3}{\alpha_1}, \quad \gamma = \frac{\lambda}{\alpha_1}, \quad A = 2\gamma\frac{V_{AC}}{V_b},$$
$$\mu = \frac{c}{\sqrt{\alpha_1}}, \quad \Omega_0 = \frac{\Omega_{ac}}{\sqrt{\alpha_1}}$$

In order to convert the governing Eq. 27 to the general form given by the Eq. 1, the nonlinear terms of Eq. 27 are expanded based on the Taylor series expansion method (Ouakad and Younis, 2010). Analysis of the obtained results confirms that expansion up to the 3rd order provides sufficient accuracy. Expanding the nonlinear terms leads to the following equation:

$$u'' + \mu u' + u(1-4\gamma) + (\beta - 14.825\gamma)u^3 = (0.831A + 2Au + 4Au^2 + 7.413Au^3)\cos(\Omega_0 \tau) \tag{28}$$

Thus, the parameters in the frequency response solution given by Eq. 15 are verified as follows:

$$\lambda^2 = 1 - 4\gamma, \qquad \xi = \beta - 14.825\gamma \tag{29}$$
$$r_0 = 0.831, \quad R = A, \quad \omega = \Omega_0, \quad r_2 = 4$$

Numerical simulations are needed to validate the analytical frequency response, given by Eq. 15 with the parameters mentioned above. To this aim, for known values of parameters, Eq. 27 is integrated numerically to obtain the steady state amplitude of the response. Then, the frequency of actuation, Ω_0 , is increased and the integration is repeated to derive the steady state response amplitude. After completing the forward frequency sweep, the frequency path is reversed and the backward frequency sweep is completed. At each step, the final state of the system is used as the initial state of the system at the next step. Figure 2, compares the obtained numerical and analytical results.

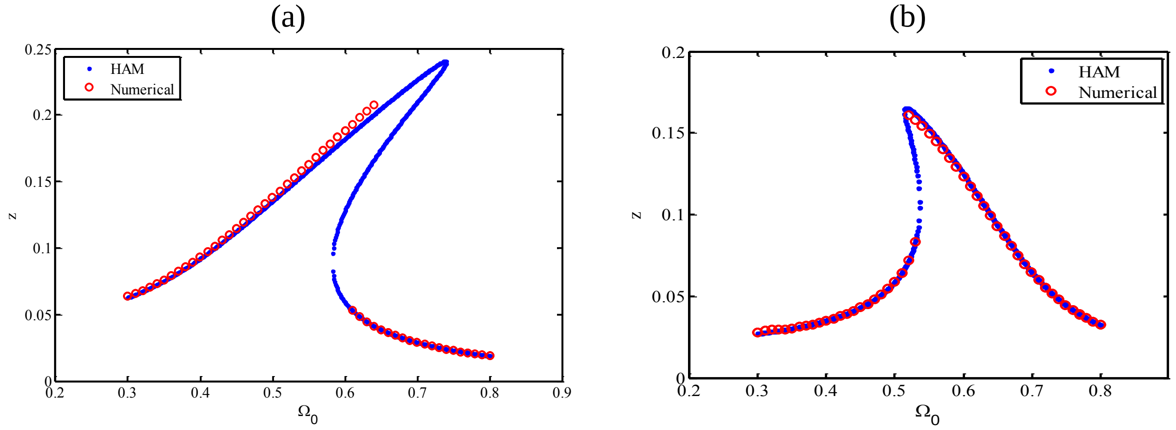

Figure2. *Numerical frequency response of the MEMS resonator with the results of the HAM solution for a)* $A = 0.01, \gamma = 0.2, \beta = 11, \mu = 0.05$ *and b)* $A = 0.01, \gamma = 0.15, \beta = -4, \mu = 0.1$

Figure 2 confirms the good agreement between the results of the numerical simulations and those given analytically by Eq. 15. This not only validates the analytical solution for the resonator's frequency response given by the Homotopy Analysis Method, but also justifies the 3rd order expansions of nonlinear terms in Eq. 27 to reach out standard form of Eq. 28. Also, Figure 2 shows that for various ranges of parameters, Eq. 15 accurately predicts the response amplitude. Furthermore, no small-parameter assumption is employed in derivation of HAM-based analytical solution. Thus, the obtained HAM solution is valid and it can be implemented for the analysis of nonlinear dynamics in engineering systems under forced vibrations and cubic nonlinearities, as in the proposed MEMS resonator.

4. CONCLUSION

In this paper, a novel analytical solution is presented to frequency response of Duffing equation with external parametric excitation using Homotopy Analysis Method (HAM). The obtained closed form equation is simple and it is not restricted to weakly nonlinear system. For verification, nano-beam under electrostatic actuation is considered a case study and comparison of analytical results with numerical ones showed that proposed analytical frequency response can capture both hardening and softening behaviors of nano-resonator with good accuracy. Since many engineering systems can be modeled by Duffing equation, the results of this paper can be useful in analyzing different engineering systems and predicting different nonlinear behaviors like hardening, softening, jump and chaos.

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